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ABSTRACT

This guide provides enrichment for students to develop tools and concepts used in various areas of mathematics. The first part presents arithmetic progressions, geometric progressions, and harmonic progression. In the second section, the concept of mathematic induction is developed from intuitive induction, using concrete activities, to the principle of mathematical induction. Logarithms constitute the third section beginning with a review of laws of exponents and bases 2, 3, and 5 and concluding with common base 10 logarithms, antilogarithms, and interpolation. The final section of the guide deals with methods for introducing the slide rule (making a slide rule, parts of the slide rule, reading numbers on the slide rule, multiplying and dividing with the slide rule). The guide contains student exercises and investigations plus suggested questions and references for teacher use. (JW)

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TOOLS AND CONCEPTS

editorial committee
Margaret Artis (1969)
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Mathematics • English • Humanities • Philosophy • Physics • Social Science • Biology • Math

THIRTEEN COLLEGE CURRICULUM PROGRAM
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Bennett College	Greensboro, North Carolina
Clark College	Dallas, Texas
Florida A and M University	Atlanta, Georgia
Jackson State College	Tallahassee, Florida
Lincoln University	Lincoln University, Pennsylvania
Norfolk State College	Norfolk, Virginia
North Carolina A and T State University	Greensboro, North Carolina
Southern University	Baton Rouge, Louisiana
Talladega College	Talladega, Alabama
Tennessee State University	Nashville, Tennessee
Voorhees College	Denmark, South Carolina

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Langston University	Langston, Oklahoma
Southern University at Shreveport	Shreveport, Louisiana
Saint Augustine's College	Raleigh, North Carolina
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Bethune-Cookman College	Daytona Beach, Florida
Grambling College	Grambling, Louisiana
Jarvis Christian College	Hawkins, Texas
LeMoyne-Owen College	Memphis, Tennessee
Southern University in New Orleans	New Orleans, Louisiana
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PREFACE

This unit is designed to provide an enrichment opportunity for students to develop tools and concepts which are used in various areas of mathematics. Arithmetic progressions, geometric progressions and mathematical induction constitute the first part of the unit. In these sections, several short investigations are described which can give students adequate introductions to these topics. The latter sections of the unit introduce students to the use of the slide rule and logarithms. Students are encouraged to make their own slide rules for both addition and multiplication. For the latter, they are encouraged to make several logarithmic slide rules using various bases. We feel that such activities give students a better understanding of the exponential principles involved.

The unit is written primarily as a teacher's guide to possible techniques for presenting the topics in the classroom, and we have used italics to indicate questions and statements that the teacher might actually use in his classroom delivery. Nevertheless, many of the investigations have been written in such a manner that they can be given directly to students for independent work. Our aim has been to write a flexible unit; thus, no order of presentation is intended with our numbering system. Our hope is that teachers will look through the investigations and exercises and select those that will aid them in the classroom.

I. PROGRESSIONS

1.1 Arithmetic Progressions

Suppose we were asked to find the sum of the first 100 consecutive numbers. We might add the numbers in pairs as follows:

1	2	3	4	.	.	.	100
100	99	98	97	.	.	.	1 .

The sum in each column is 101, and since there are 100 columns, the total of all these sums is 100 x 101 or 10,100. *Is this the required sum?* The instructor should have the students work with these sums until they have convinced themselves that in the process of adding the totals in the 100 columns, we obtain double the sum 1 + 2 + . . . + 100. This tells us that the required sum is 5,050.

Can you use this process to get the sum of the first 150 consecutive numbers? The first 200? Can you go through this procedure to get the sum of the first n *consecutive numbers for any* n?

The students with guidance from the instructor should see that their procedure for the sum of the first n numbers takes the following form:

1	2	3	.	.	.	n
n	n - 1	n - 2	.	.	.	1 .

The sum in each column is $n + 1$, and since there are n columns, the required sum is $n(n + 1)/2$. *Does this agree with the previous examples in which* $n = 100$, $n = 150$, $n = 200$?

Next ask the students to examine the sum: $2 + 5 + 8 + \ldots$ and try to determine the nth term and the sum of the first n terms. The instructor should point out that the sum can be written,

$$(1) \quad 2 + (2 + 3) + (2 + 2 \cdot 3) + \ldots ,$$

and this should be enough for them to figure out that the nth term is $2 + (n - 1)3$. To get at the sum of the first n terms in (1), the instructor might lead the students to see the following:

$$2 + (2 + 3) + (2 + 2 \cdot 3) + \ldots + [2 + (n - 1)3] =$$

$$\underbrace{2 + 2 + \ldots + 2)}_{\text{n times}} + [3 + 2 \cdot 3 + \ldots + (n - 1)3] =$$

$$2n + 3(1 + 2 + \ldots + n - 1).$$

Can you write a short expression for $1 + 2 + \ldots + n - 1$?

Hopefully they will be able to use the formula developed above to write $n(n - 1)/2$ for this expression, and to see that

$$2 + (2 + 3) + (2 + 2 \cdot 3) + \ldots [2 + (n - 1)3] =$$

$$2n + \frac{3[n\ (n - 1)]}{2} .$$

Now the students should be asked to find the sum of the first n terms in several other examples such as

$$4 + 6 + 8 + \ldots$$

$$3 + 7 + 11 + 15 + \ldots$$

and to see what similarities they can find in the several examples. They should be able to figure out that if we let a denote the first term, the terms in the examples have the form

$$(2) \quad a,\ a + d,\ a + 2d,\ \ldots,\ a + (n - 1)d,\ \ldots$$

where d is the difference between any two consecutive terms. We call (2) an arithmetic progression; and if only n terms are considered, the "last" term $\ell = a + (n - 1)d$. Generalizing on the examples, they should be able to write the sum S of the first n terms as:

$$(3) \quad S = an + \frac{dn(n - 1)}{2} = \frac{n}{2}\,[2a + (n - 1)d]\ .$$

Now ask the students if they can show that this expression for S is the same as:

$$(4) \quad S = \frac{n}{2}\,(a + \ell).$$

As a final check, tell the students to test formulas (3) and (4) on all of the examples that have been considered previously.

CHALLENGE 1: What is the 20th term and the sum of the first 16 terms of the arithmetic progression 2, 7, 12, . . . In this instance, 20 represents the last term that you are concerned with in the arithmetic progression. Hence, $\ell = a + (n - 1)d = 2 + (20 - 1)5 = \boxed{97}$ the 20th term. For the sum, S, of the first 16 terms of the arithmetic progression, we use (3):

$$S = 16/2\ [2(2) + (16 - 1)5] = \boxed{632}$$

What is the 39th term, and the total of the first 39 terms of the arithmetic progression 10, 4, -2, -8, . . . In this case, $a = 10$, $d = -6$, $n = 39$. Thus we have:

$$\ell = 10 + (39 - 1)(-6) = \boxed{-218}$$

Substituting in (4) gives:

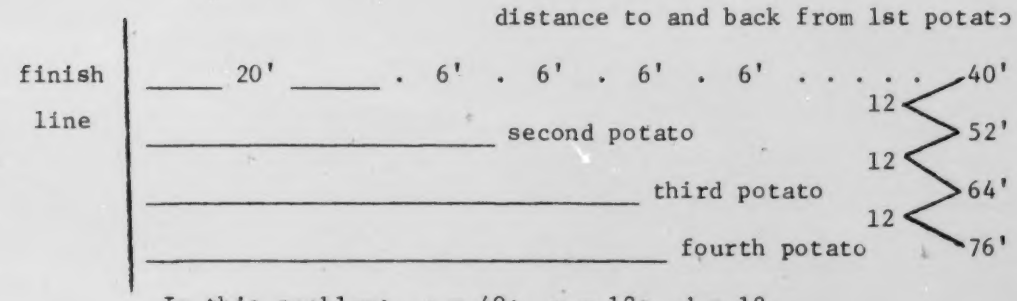
$$S = 39/2\,[10 + (-218)] = \frac{39 \cdot (-208)}{2} = \boxed{-4056}$$

CHALLENGE 2: The distance in feet through which a body falls from rest in t seconds is (approximately) $16t^2$. Show that the distances through which it falls during the first, second, third, . . . seconds form an arithmetic progression. How far will the body fall in the tenth second? How far will it fall in the first 20 seconds?

during	substitution	feet	difference
1st"	$16(1)^2$	16	
2nd"	$16(2)^2 - 16(1)^2$	48	32
3rd"	$16(3)^2 - 16(2)^2$	80	32
4th"	$16(4)^2 - 16(3)^2$	112	32
.	.		
.	.		
.	.		
10th"	$16(10)^2 - 16(9)^2$	304	

From the chart we see that the body falls 304 feet during the tenth second. To find out how far it falls during the first twenty seconds we plug into Formula (3): $S = 20/2\ [2(16) + (20 - 1)32] = 6400$ feet.

CHALLENGE 3: In a potato race, 12 potatoes are placed 6 feet apart in a straight line. If the potatoes are to be picked up and returned to the finish line one at a time, and if the first potato is 20 feet from the finish line, what distance will be covered by a contestant who finishes the race?



In this problem: $a = 40$; $n = 12$; $d = 12$.

Distance to last potato is:

$$\ell = a + (n - 1)d = 40 + (12 - 1)12 = 172'$$

Total distance for contestant who finishes race is:

$$S = n/2\ (a + \ell) = 12/2\ [40 + 172] = 6 \cdot 212 = 1272'$$

1.2 Geometric Progressions

Ask the students to give the sum of the first ten numbers in the sequence:

(1) 3, 6, 12, 24, 48, 96, . . .

After they have found the sum, ask them if their answer agrees with the answer obtained as follows. First write the sum S of the first ten terms of (1) in this way:

$$S = 3 + 3\cdot 2 + 3\cdot 2^2 + \ldots + 3\cdot 2^9.$$

What do you get when you substract 2S from S?

$$S - 2S = (3 + 3\cdot 2 + 3\cdot 2^2 + \ldots + 3\cdot 2^9) - 2(3 + 3\cdot 2 + \ldots + 3\cdot 2^9)$$

$$= 3 - 3\cdot 2^{10}.$$

Thus,

$$S(1 - 2) = 3 - 3\cdot 2^{10}$$

and

$$S = \frac{3 - 3\cdot 2^{10}}{1 - 2}$$

Now ask the students if they can figure out how to obtain the sum of the first 15 terms in (1), and ultimately the first n terms. These examples should suggest to them the formula:

$$S = \frac{3 - 3\cdot 2^n}{1 - 2} = \frac{3(1 - 2^n)}{1 - 2}$$

for the sum of the first n terms.

Next have the students try out their procedure on other sequences such as:

2, 8, 32, 128, . . .

3, 15, 45, 135, . . .

and ask them for similarities among all of the examples. They should see that all of the sequences have the form:

(2) $a, ar, ar^2, \ldots, ar^{n-1}, \ldots$

where r is the "common ratio" between consecutive terms. We call (2) a geometric progression. If only n terms are considered, the last term is given by:

(3) $\ell = ar^{n-1}$

and the sum of the first n terms is

(4) $S = \frac{a(1 - r^n)}{1 - r}$.

CHALLENGE 1: If a student undertakes to deposit 1¢ on October 1, 2¢ on October 2, 4¢ on October 3, 8¢ on October 4, etc., how much will he deposit in his account from October 1 through October 15? If he continued to deposit through October 30, how much would he deposit on that day?

In this instance, $a = .01$, $r = 2$, $n = 15$. Substituting into (4) gives: $S = \frac{.01\ (1 - 2^{15})}{1 - 2} = \$327.67$

For October 30, substituting into (3) gives:

$$\ell = .01 \cdot 2^{29} = \$5,368,709.12$$

CHALLENGE 2: In a certain colony of bacteria each divides into 2 every hour. How many will be produced from a single bacillus if the rate of division continues for 12 hours?

In this situation, $a = 2$, $r = 2$, $n = 12$. (Why doesn't $a = 1$?) Substituting into (3) gives:

$$\ell = 2 \cdot 2^{11} = 4096.$$

CHALLENGE 3: A rubber ball is dropped from a height of 81'. Each time it strikes the ground it rebounds 2/3 of the distance through which it last fell. What distance has the ball traveled when it strikes the 6th time? The required distance is the sum of the first 6 falls and the first 5 rebounds. For the falls: $a = 81$, $r = 2/3$, $n = 6$. Thus:

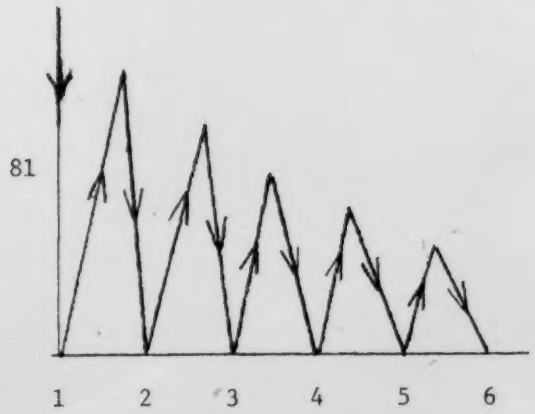
$$S = \frac{81\ [1 - (2/3)^6]}{1 - 2/3} = 665/3 \text{ feet.}$$

Rebounds: $a = 2/3(81) = 54$, $r = 2/3$, $n = 5$. Thus,

$$S = \frac{54\ [1 - (2/3)^5]}{1 - 2/3} = 422/3 \text{ feet.}$$

Total distance traveled = 665/3 feet + 422/3 feet =

665/3 feet + 422/3 feet = 362 1/3 feet.



CHALLENGE 4: Find the sum of the first 8 fractions in the list:

3/10, 3/100, 3/1000, . . .

Is this a geometric progression? If so, what is the common ratio r?

1.3 Harmonic Progression

A set of numbers such as {2, 2/3, 2/5, ...} is a harmonic progression if the reciprocals of the terms in the set form an arithmetic progression.

CHALLENGE 1: Show that the above set of numbers is a harmonic progression and find term 30. First form reciprocals: 1/2, 3/2, 5/2, ... Substituting in $\ell = a + (n - 1)d$, where $a = 1/2$, $d = 1$, $n = 30$, we have:

$$\ell = 1/2 + (30 - 1) = 1/2 + 58/2 = 59/2.$$

Thus the 30th term of the harmonic progression is 2/59.

CHALLENGE 2: The harmonic mean between 2 and 4 is defined to be that number x which satisfies the equation

$$\frac{1}{x} = \frac{1/2 + 1/4}{2}.$$

State in words what the harmonic mean between 2 and 4 is. What is its numerical value?

CHALLENGE 3: Prove that the harmonic mean between any two numbers a and b is $\frac{2ab}{a+b}$. From Challenge 2, we seek a number x such that

$$\frac{1}{x} = \frac{1/a + 1/b}{2}.$$

Now show that when we solve this equation for x we get $x = \frac{2ab}{a+b}$.

1.4 Infinite Geometric Progressions

A. Write the numbers .9, .99, .999, . . . on the board and ask the students to guess what number we will be getting close to if the list is continued. They will probably guess the answer 1. In order to get them to think about a "proof" of this fact, the instructor might write

$$.999\ldots = .9 + .09 + .009 + \ldots$$

and ask: *How are the terms on the right related to each other?* If these terms are written as fractions:

(1) $9/10 + 9/100 + 9/1000 + \ldots$

perhaps they will see that what is involved is the sum of the terms of a geometric progression with $a = 9/10$ and $r = 1/10$. (The terms in (1) form an infinite geometric progression.) Thus the decimals .9, .99, .999, . . . are actually sums of geometric progressions ("partial sums" of the fractions in (1)). Using the formulas of section 1.2, we get for the sums of these fractions:

$$.9 = \frac{9/10\ (1 - 1/10)}{1 - 1/10}$$

$$.99 = \frac{9/10\ (1 - 1/100)}{1 - 1/10}$$

$$.999 = \frac{9/10\ (1 - 1/1000)}{1 - 1/10}\ , \text{ etc.}$$

a. What will the fourth sum be? The tenth?

b. *What will the nth sum be? What does* $\underbrace{.999\ldots9}_{\text{n digits}}$ *equal?*

c. *How do these sums differ from each other?*

d. *What number are these sums getting close to as the number of fractions added increases?*

The students will probably figure out that the quantity $1 - (1/10)^n$ in the numerators is getting closer and closer to 1 as n increases, so the answer to question c is:

$$\frac{9/10}{1 - 1/10} = 1.$$

e. *What does the infinite decimal expansion .999 ... equal? Is it a rational number?*

B. The students may now be asked the question: *What do you suppose the inifinite decimals .888..., .777..., .111..., equal?* They should use the above procedure to get the answers:

$$\frac{8/10}{1 - 1/10} = 8/9$$

$$\frac{7/10}{1 - 1/10} = 7/9$$

$$\frac{1/10}{1 - 1/10} = 1/9.$$

How about the fractional equivalent of .kkk..., where k is any positive integer?

C. Now see if the students can extend their results to investigate decimals such as .101010... or .123123.... Notice for example that the first of these decimals can be written as the sum of the infinite progression 1/10, 1/1000, 1/100000, ... The instructor may take this opportunity to discuss in general the sums of infinite geometric progressions $a + ar + ar^2 + \ldots$, and to have the students investigate whether the partial sums always approach the number $a/(1 - r)$.

EXERCISES

1. Find the sum of the first million positive integers.
2. The contract on a construction job specifies that the job must be completed by October 30, 1971. If not, a penalty of $200 will be charged for the first day of delay of completion, in addition, $250 will be charged for the second day, in addition to that, $300 for the third day, etc. How much in penalties will the contractor have to pay if the job is not completed until December 1, 1971?
3. At a raffle 200 tickets are marked 5, 10, 15, 20, ..., and are drawn at random by those people entering. If the price of each ticket is the number of cents marked on each ticket, how much money will be taken if all of the tickets are sold?
4. If the value of the articles to be raffled in exercise 3 was $100, what is the smallest number of tickets that must be sold so as to insure no loss to the people holding the raffle?
5. John Smith's great, great, great, great grandfather was named Hannibal. How many of John's foreparents (at a maximum) came after Hannibal's generation?
6. A farmer wishes to have his horse shod. He offers to pay a blacksmith either $125 per shoe or 1 cent for the first nail, 2 cents for the second nail, 3 cents for the third nail, etc., for the 4 shoes. Each shoe requires 8 nails. Which is the better deal for the blacksmith?
7. Write the following decimals as single fractions:

   .767676...
   .1001001001...
   3.616161...

II. MATHEMATICAL INDUCTION

In this section we introduce the concept of proof by mathematical induction. We first deal with investigations which intuitively require "induction" in their solutions, and this leads to the statement of the principle of mathematical induction. Then we examine a number of problems and discuss their solutions using mathematical induction. The section deals exclusively with problems and formulas involving the positive integers, thus when the word "number" is used in this section, we shall mean "counting number."

2.1 Intuitive Induction

(1) If we have some dots on paper, these dots are called linear, if it is at all possible for a straight line to be drawn through all of them. If such a line cannot be drawn, the dots are nonlinear in arrangement.

Figure 1

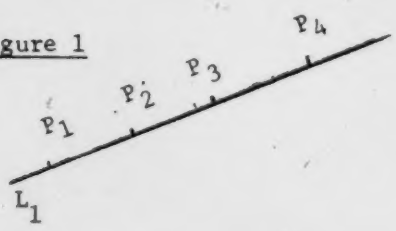
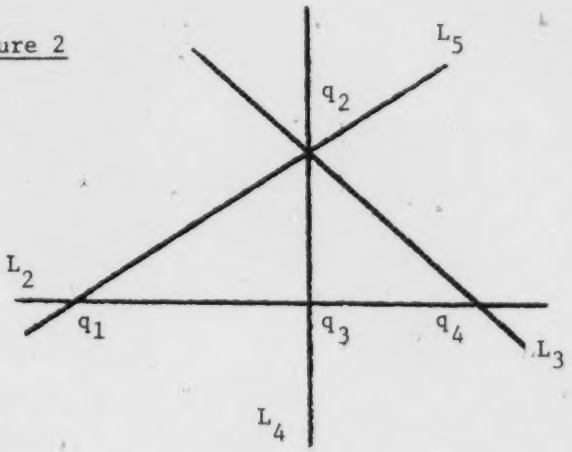


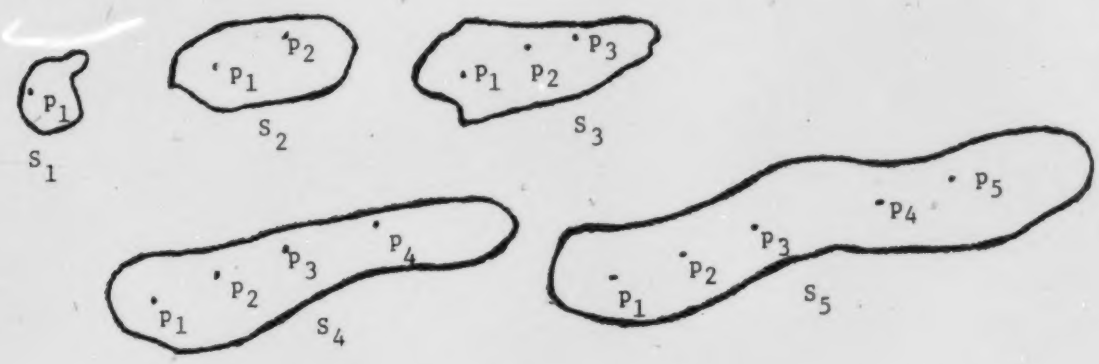
Figure 2



In Figure 1, the dots p_1, p_2, p_3 and p_4 are linear since line L_1 is drawn through all four of these dots. We can draw a line L_2 through the dots q_1, q_3 and q_4 of Figure 2, but this line does not go through q_2. Similarly, line L_3 can be drawn through q_2 and q_4, but it does not go through q_1 and q_3. Line L_4 goes through q_2 and q_3, but not through q_1 or q_4. Line L_5 goes through q_1 and q_2 but not through q_3 or q_4. In general, no line can be drawn through all four dots q_1, q_2, q_3 and q_4, and hence these dots are nonlinear.

Let us now suppose we have an infinite set of dots p_1, p_2, p_3, p_4, . . ., p_n, . . . Let S_1 be the set consisting of the first dot p_1. Let S_2 be the set of the two dots p_1 and p_2; let S_3 be the set of the first three dots, p_1, p_2, p_3; and in general, for each positive integer n, let S_n be the set consisting of precisely the first n dots of our infinite set (Figure 3).

Figure 3



Try to answer the following questions (keep a detailed list of your reasons).

a. *Is* S_1 *linear or nonlinear? What about* S_2*?*

b. *If it is found that* S_1, S_2, S_3, S_4 *and* S_5 *are linear, what can you say about* S_6*?*

c. *If it is found that* S_1, S_2 *and* S_3 *are linear and* S_4 *is nonlinear, what can you say about* S_5*?*

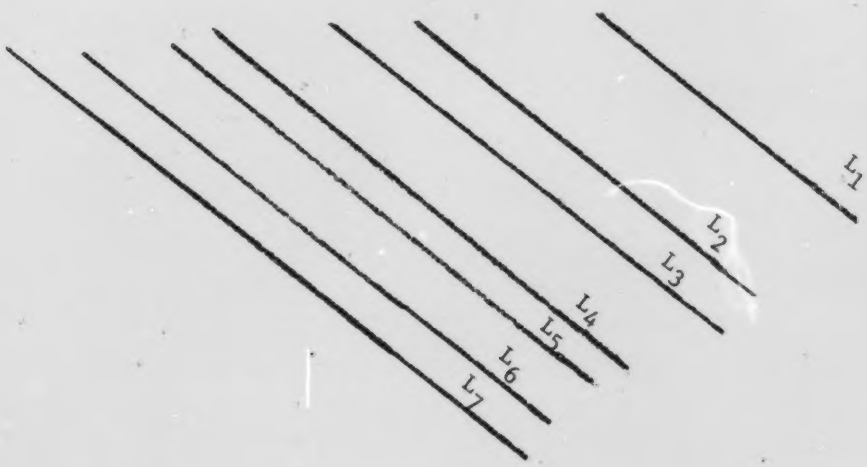
d. *Can* S_i *be linear for each integer* i*?*

e. *Suppose* S_3 *is nonlinear. What can you conclude about* S_i *where* $i > 3$*?*

(2) A set of lines is said to be a <u>parallel system</u> if no two lines in the set cross each other. See Figure 4, for example.

Suppose we have an infinite set of lines L_1, L_2, L_3, . . ., L_n, . . .

<u>Figure 4</u>



Let S_1 be the first line L_1; let S_2 be the first two lines L_1 and L_2; and in general, for any positive integer n, let S_n be the set of lines consisting precisely of the first n lines of the infinite set. *Examine the following questions:*

a. *Is S_1 a parallel system?*

b. *If it is found that S_1, S_2, S_3 and S_4 are parallel systems, what can you say about S_5?*

c. *If L_1 and L_2 are known to cross, what can you say about S_i for each integer i?*

Give explanations for your reasoning. Can you give another explanation that is essentially different from one just given?

(3) A person is said to be of African descent if either his father was born in Africa or was of African descent. In other words, being of African descent is an inherited property. Let S_1, S_2, S_3, S_4, . . ., be a sequence of men in which for each integer i, S_{i+1} is the son of S_i. Is S_{10} of African descent? *Discuss some minimum requirements for S_i to be of African descent, for all integers i. Discuss other possibilities.*

(4) The decimal expansion for the fraction 4/7 begins this way:

.5714 ...

a. *What is the fifth digit in the expansion? What is the sixth? What is the seventeenth?*

b. *Knowing the 17th digit, could you predict the 18th?*

c. *Can you find a way of telling what the nth digit in the expansion is for any n?*

Exercise: *Investigate the decimal expansions of other fractions in this same manner.*

(5) Consider the sequence: 1/2, 1/4, 1/6, . . .

a. Can you continue this sequence to 10 terms?

b. What is the 20th term?

c. What is the nth term?

(6) Answer the questions in (5) for the sequence:

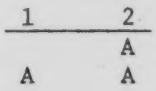
2, 3, 5, 7, 11, 13, . . .

(7) Answer the following questions.

a. What is the first odd number?

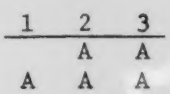
How many A's are there above?

b. What is the second odd number?



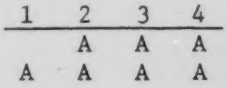
How many A's are there above?

c. What is the third odd number?



How many A's are there above?

d. What is the fourth odd number?



How many A's are there above?

e. What is the fifth odd number?

1	2	3	4	5
	A	A	A	A
A	A	A	A	A

How many A's are there above?

f. What is the tenth odd number?

1	2	3	4	5	6	7	8	9	10
	A	A	A	A	A	A	A	A	A
A	A	A	A	A	A	A	A	A	A

How many A's are there above?

g. What is the 20th odd number?

How many A's should there be?

h. What is the 88th odd number?

How many A's should there be?

i. What is the nth odd number?

How many A's should there be?

(8) Give each student a geo-board and rubber bands.

a. In the first column at the bottom of the geo-board put a rubber band around the first odd number of pins.

b. In the second column at the bottom put a rubber band around the second odd number of pins, and continue this in the third and fourth columns.

c. If this process is contined, how many pins will you cover in the fifth column? Sixth column? Can you guess how many would be covered in the tenth column? How about the nth column? Can you guess a formula or expression which will give the nth odd number?

d. Does your formula correctly give the first odd number? The second? The fourth? When you went from the fourth column to the fifth, how many more pins did you have to cover? Is your formula correct for the fifth odd number? When you went from the fifth column to the sixth, how many more pins did you have to cover? Is your formula correct for the sixth odd number?

e. What is the 15th odd number? How do you get from the 15th to the 16th odd number? Is this what your formula gives for the 16th odd number? What does your formula give for the kth odd number? How do you get from the kth to the (k + 1)st odd number? Is this what your formula gives for the (k + 1)st odd number?

f. What can you conclude about your formula?

Exercise: Go through a similar investigation with the even numbers. Find a formula for the nth even number and test the formula in the manner above.

Exercise: Investigate the multiples of three similarly.

2.2 Mathematical Induction

(1) Sum of the odd numbers

Give each student a geo-board and several rubber bands.

a. *In the first column at the bottom cover the first odd number of pins.*

b. *Put another rubber band on the geo-board so that it covers the second odd number of uncovered pins closest to the pin already covered. How many pins have you covered so far?*

c. *Now put another rubber band on the geo-board so that it covers the first pin and the third odd number of uncovered pins which are closest to the pins already covered? What is the total number of pins now covered? What does this number represent?* Answer: the sum of the first three odd numbers.

d. *Can you use the geo-board to find the sum of the first four odd numbers? First five? Can you guess the sum of the first ten odd numbers? Can you guess a formula which will give the sum of the first* n *odd numbers? Check your formula for several values of* n.

e. *If we go from the sum of the first five odd numbers to the sum of the first six, how many pins do we have to add? How many do we have to add in going from the sum of the first six to the sum of the first seven? Seven to eight?*

f. *What does your formula give for the sum of the first k odd numbers? What do we have to add to this to get the sum of the first k + 1 odd numbers? What is the sum of the first k + 1 odd numbers? Does this agree with what your formula says the sum of the first k + 1 odd numbers is?*

g. *Do you have any conclusions about your formula?*

At this point the instructor might refer to the type of reasoning that occurred in the previous examples as reasoning by mathematical induction in which we try to argue that a statement or formula is true for all of the counting numbers. First we prove that the formula is true for the first few numbers. Second, we prove that the truth of the formula is a kind of inheritance property, namely, that if the formula holds for any given number k, then it also holds for the very next number, k + 1. *Is it reasonable to conclude that a formula or statement is true for all of the counting numbers if these two steps can be verified for the formula? Why?*

(2) Triangular Numbers

Using the peg board, (the geo-board could also be used) begin by asking the students to construct the smallest simple plane geometric figure. They will probably respond by inserting one peg.

a. *Move to an adjacent column and add the least number of additional pegs so that in this new column there is one more peg than in the previous column. Use a rubber band to surround the pegs. What figure do you have? How many pegs are enclosed?*

The students will get a three-peg triangle, and the instructor should refer to 3 as the second triangular number, implying that 1 is the first triangular number.

b. *Move to the next column and form the next larger triangle similar to the one before. How many pegs do you need to use here? What is the relationship between the number of pegs in this column and the number in the previous column? What is the third triangular number?*

c. *Find the fourth, fifth, sixth, etc., triangular numbers in the manner described above. Can you guess what the tenth triangular number is? How about the 20th? How many pegs will be in the 10th column, 20th column,* nth *column? Can you write an expression or formula for the* nth *triangular number?*

We expect the students to be able to figure out that there are n pegs in column n and that the nth triangular number is $1 + 2 + \ldots + n$. If he wishes the instructor might use this as an opportunity to "walk through" a kind of induction proof of this fact. If the students are familiar with arithmetic progressions, they might be asked questions such as:

d. *What kind of progression or sequence do you have when you consider the numbers (of pegs) in each successive column? What is the sum of the numbers in each column and the numbers of all columns prior to it? Can you now write a formula for the* nth *triangular number,* $1 + 2 + \ldots + n$?

They can use the formula for the sum of an arithmetic progression to write $1 + 2 + \ldots + n = n(n + 1)/2$, and the instructor might ask for a proof of this by mathematical induction (see section 2.3). An alternate method of motivating this formula is as follows:

e. *On the peg board construct the triangle as above, which corresponds to the second triangular number. Now construct a rectangle which encloses twice as many pegs as the triangle. How many pegs are enclosed in the rectangle? Make a rectangle enclosing twice as many pegs as the triangle which corresponds to the third triangular number. How many pegs are enclosed? How many are enclosed in the fourth rectangle (rectangle corresponding to the fourth triangular number)? The fifth? Can you guess how many pegs would be enclosed in the 15th rectangle? How about the* nth? ANSWER: *Probably* $n(n + 1)$. *What is the* nth *triangular number? Recalling how we constructed the above rectangles can you now guess what a shorter expression for* $1 + 2 + \ldots + n$ *might be?* ANSWER: *probably* $n(n + 1)/2$. *Test, by mathematical induction, to see if* $1 + 2 + \ldots + n = n(n + 1)/2$ *for all* n.

(3) Square Numbers

a. *Use the peg board to construct the triangle which corresponds to second triangular number. How many more pegs will you need to add to the figure to make a square? Call the number of pegs enclosed in the square the second square number. What do you think the first square number is?*

b. *Can you determine the third, fourth and fifth square numbers from the third, fourth and fifth triangular numbers? How about the 15th square number from the 15th triangular number? Can you write an expression for the* nth *square number?*

c. *Can you write a formula relating the* nth *triangular number to the* nth *square number? Can you prove your formulas by mathematical induction?*

(4) Rectangular Numbers

Using the peg board have the students to construct the smallest non-degenerate line figure. They will probably insert two pegs and get a figure such as: . .

a. *Using the least number of pegs, construct a non-degenerate rectangle (not a square) which includes the previous two pegs. How many pegs did you use? Call this number the second rectangular number, 2 being the first rectangular number.*

b. *Using the least number of pegs, construct a rectangle so that each dimension of the previous rectangle is increased by one. How many pegs are enclosed in this rectangle? Call this number the third rectangular number.*

c. *Can you find the fourth, fifth and sixth rectangular numbers? What is the 25th rectangular number? The* nth*? What is a relationship between the triangular numbers and the rectangular numbers?*

(5) Pentagonal Numbers

Reproduce the following patterns and distribute them to the class:

1 5 12 22

a. *The numbers 1, 5, 12, 22, . . . are called pentagonal numbers. Can you figure out what the next few pentagonal numbers are? Can you guess what the 20th pentagonal number is? How about the* nth*?*

b. *Can you express each pentagonal number as the sum of a square number and another number? What can you say about the other number? Can you find a relationship between pentagonal numbers and triangular numbers? Can you now write a formula which will give the* nth *pentagonal number? Can you prove your formulas by mathematical induction?*

(6) Decimal Exercise

Examine the following decimal expansion: .101001000...

a. *What are the next 5 digits in the expansion?*

b. *Can you guess what the 19th digit will be? Could you predict the 34th digit?*

c. Can you devise a scheme whereby you can tell what the nth digit is for any n?

d. Can you write a formula for the nth digit and verify it with mathematical induction?

HINT: Make a list showing which terms in the expansion are 1's. The first, the third, . . . are 1's, so your list should begin: 1, 3, . . .

e. Is the sequence familiar to you? See investigation (2).

f. Can you write a formula giving the nth term of the sequence?

g. Can you now answer question c? How about question d?

h. Is the number .101001000... a rational number? Can you use the procedure in section 1.4 to determine what this number is?

(7) Sum of the Even Numbers

Consider the sequence: 2, 6, 12, 20, 30, 42, . . .

Exercise: Find a formula or expression which will give the nth term of the sequence. Use the geo-board or the peg board, to devise a shorter formula as in (2) above. Use mathematical induction to verify your formulas.

HINT: Using the geo-board,

a. Cover the bottom two pins in the first column with a rubber band.

b. Put another rubber band on the geo-board so that it touches the four pins which are closest to the pins already covered. How many pins now have rubber bands touching them?

c. *Put another rubber band on the geo-board so that it touches the six pins which are closest to the pins already covered. How many pins now have rubber bands touching them?*

d. *Put another rubber band on the geo board so that it touches the eight pins which are closest to the pins already covered. How many pins now have rubber bands touching them?*

e. *What can you say about the sequence of numbers that you have generated? How are they related to the given sequence?*

(8) Sequence Exercise

Consider the sequence: 3, 8, 15, 24, . . .

a. *What is the fifth term of this sequence? What is the 19th term?* nth *term?*

b. *How do you go from term to term in this sequence?*

c. *Does your expression for the* nth *term equal* $n(n + 2)$*? Is this true for all* n*? Can you prove it using mathematical induction?*

(9) Factoring $x^n - 1$

a. *What are the factors of the polynomial* $x^2 - 1$*?*

ANSWER: $x^2 - 1 = (x - 1)(x + 1)$.

b. *What can you do to the left side of the latter equation so that it becomes* $x^3 - 1$*?*

ONE ANSWER: *multiply by* x*, then add* x*, and then subtract* 1.

c. How do these same operations affect the right side of the equation?

ANSWER: the right side becomes:

$x(x - 1)(x + 1) + x - 1.$

d. Can you factor this last expression? Can you now write the factors of $x^3 - 1$?

e. Knowing the factors of $x^3 - 1$, can you factor $x^4 - 1$?

HINT: How did you get the factors of $x^3 - 1$, knowing them for $x^2 - 1$?

f. What are the factors of $x^5 - 1$? Can you guess what the factors of $x^n - 1$ are?

g. If you knew the factors of $x^k - 1$, could you write the factors of $x^{k+1} - 1$?

h. Use mathematical induction to prove that your formula for the factors of $x^n - 1$ is correct for all n.

(10) Base Two Exercise

a. How many non-zero one-digit numbers are there in base two?

b. How many non-zero base two numbers have at most two digits?

c. How many non-zero base two numbers have at most three, four, five digits?

d. Can you guess a formula which will give the number of non-zero numbers having at most n digits in base two?

e. Can you prove your results using mathematical induction?

ADDITIONAL CHALLENGE: Can you find any connection between examples (9) and (10)? Can you write a shorter expression for your formula in part d of (10)?

2.3 Illustrative Examples

(1) In this example we illustrate the use of mathematical induction to prove:

a. $1 + 2 + \ldots + n = n(n + 1)/2$ is true for all $n(n > 0)$.

First, we see if the equation holds for some values of n.

If $n = 1$: $1 = 1(1 + 1)/2 = 1$.

If $n = 2$: $1 + 2 = 2(2 + 1)/2 = 3$.

If $n = 3$: $1 + 2 + 3 = 3(3 + 1)/2 = 6$.

Try several more values of n *to convince yourself that the equation is true.* Now we assume that the equation is true when $n = k$. (k represents an arbitrary positive integer.)

b. If $n = k$: $1 + 2 + \ldots + k = k(k + 1)/2$.

Next we want to prove the truth of our equation for $n = k + 1$. By substituting $k + 1$ for n in a, above, we will see what we must prove.

c. For $n = k + 1$: $1 + 2 + \ldots + k + (k + 1) = (k + 1)\ (k + 2)/2$

How can we obtain c *from* b? To get from the left side of equation b to the left side of equation c, we add $k + 1$. *Will adding* $k + 1$ *to the right side of* b *give the right side of* c? Yes, since:

$$k(k + 1)/2 + (k + 1) = (k^2 + 3k + 2)/2 = (k + 1)\ (k + 2)/2.$$

We have shown that the formula in a holds for the first few positive integers and that if it is true for one positive integer it is true for the succeeding one. Thus, according to the principle of mathematical induction, it is true for all of the positive integers.

(2) Consider the following partial sums:

$S_1 = 3$

$S_2 = 3 + 5 = 8$

$S_3 = 3 + 5 + 7 = 15$

$S_4 = 3 + 5 + 7 + 9 = 24$

$S_5 = 3 + 5 + 7 + 9 + 11 = 35$

By trial and error guessing or referring to investigation (8) in section 2.2 above, we suspect that the nth partial sum, $S_n = n(n + 2)$. *(Can you find a way to convince yourself of this using the geo board?)* Let us prove this by mathematical induction.

Notice that the numbers on the left sides of the equations above are odd numbers and therefore can be represented by $2n - 1$ or by $2n + 1$. Since we want the first odd integer in our list to be 3 (when $n = 1$), we choose the representation, $2n + 1$. Thus, our problem is to prove that for all $n \geq 1$,

a. $3 + 5 + 7 + \ldots + 2n + 1 = n(n + 2)$.

A quick check of the partial sums listed above shows that equation a is true for $n = 1, 2, 3, 4$ and 5. Assume the equation holds for an arbitrary number k, i.e.,

b. $3 + 5 + 7 + \ldots + 2k + 1 = k(k + 2)$.

Now we wish to use b to prove that the equation holds for $n = k + 1$, i.e., to prove:

c. $3 + 5 + 7 + \ldots + 2k + 1 + [2(k + 1) + 1] = (k + 1)\ [(k + 1) + 2]$

$= (k + 1)\ (k + 3)$.

By adding $2(k + 1) + 1$ to both sides of b we obtain:

$$3 + 5 + 7 + \ldots + 2k + 1 + [2(k + 1) + 1] = k(k + 2) + 2(k + 1) + 1$$

And since,

$$k(k + 2) + 2(k + 1) + 1 = k^2 + 4k + 3 = (k + 1)\ (k + 3),$$

we have shown that our equation holds for $n = k + 1$, assuming it holds for $n = k$. Hence, the formula is true for all $n \geq 1$ by mathematical induction.

EXERCISES

1. We have seen that $1 + 2 + 3 + \ldots + n = 1/2\ n(n + 1)$ for all $n \geq 1$. Can you use this fact to complete the following equations?

   $2 + 4 + 6 + \ldots + 2n = ?$

   $3 + 6 + 9 + \ldots + 3n = ?$

   $4 + 8 + 12 + \ldots + ? = ?$

   $5 + 10 + 15 + \ldots + ? = ?$

   .
   . .
   .

   $k + ? + ? + \ldots + ? = ?$, k any positive integer.

   Prove your guesses by mathematical induction.

2. What is the maximum number of line segments a line can be divided into with 50 points? n points?

3. What is the maximum number of sections a plane can be divided into with 50 lines? n lines?

4. If it is given that the sum of the angles of any triangle is 180°, use mathematical induction to show that the sum of the angles of an n-sided figure is $(n - 2)180°$.

5. The number of diagonals in a convex polygon of n sides is $1/2n\ (n - 3)$.

6. Determine experimentally a representation of the value of:

   $$1/(1)(2) + 1/(2)(3) + 1/(3)(4) + \ldots + 1/(n)(n+1)$$

   for any given integer n. Can you prove your result by mathematical induction?

7. For the expressions $(a + b)^0$, $(a + b)$, $(a + b)^2$, etc., we get 1, $a + b$, $a^2 + 2ab + b^2$, etc. We can arrange the coefficients in the following array, called Pascal's triangle:

   ```
          1
        1   1
      1   2   1
    1   3   3   1
   ..............
   ```

   What conclusion can you draw about obtaining the next set of coefficients when the previous set is known? Use this conclusion to determine the next two rows. What can you say about the sum of the coefficients of each expansion? Prove this by mathematical induction.

8. Prove that $1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$, using mathematical induction.

9. Prove using mathematical induction, that $a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} = a(1 - r^n)/(1 - r)$.

10. Prove, using mathematical induction, that $4 + 3 + 2 + 1 \ldots + (5 - n) = 1/2n\ (9 - n)$.

11. Prove by mathematical induction that $x^n - 1 = (x - 1)(1 + x + \ldots + x^{n-1})$.

12. Can you use exercise 11to guess a formula for $x^n - y^n$? Prove it by mathematical induction.

13. Find a formula which tells the number of non-zero numbers having at most n digits in base 2.

14. Can you extend your results of problem 13 to base 3? base k?

15. How many subsets are there in a set with 20 members? n members?

III. LOGARITHMS

3.1 Introduction

In this section it is hoped that the instructor will lead the students to discover the theorems and uses of logarithms.

In 1614, the English mathematician Napier developed logarithms for the specific purpose of reducing the work involved in multiplying and dividing large numbers.

Quite apart from their use in computation, logarithms are indispensible in nearly all branches of mathematics, science and engineering. Before we begin our discussion of logarithms, however, we must first mention exponents and their role in multiplying and dividing numbers.

A. Review of Exponents

Several exercises of the following types can be used:

1. Is there a better way to write $B \cdot B \cdot B \cdot B \cdot B$?
2. Multiply: $A^5 \cdot A^4 = A \cdot A \cdot A \cdot A \cdot A \cdot A \cdot A \cdot A \cdot A = ?$

   $A^6 \cdot A^2 = A^8$; $A^3 \cdot A^4 = A^7$; and so on. All of these examples show the rule $A^x \cdot A^y = A^{x+y}$.
3. Divide: $\frac{X^5}{X^2} = \frac{X \cdot X \cdot X \cdot X \cdot X}{X \cdot X} = ?$ $\qquad \frac{X^{20}}{X^4} = X^{16}$

   The above examples show the rule $\frac{x^{A+B}}{x^B} = x^A$.
4. There is also the rule of powers of powers of numbers.

   $(2^3)^5 = 2^3 \cdot 2^3 \cdot 2^3 \cdot 2^3 \cdot 2^3 = 2^{15}$

   Or more abstractly, the rule $(2^A)^B = 2^{A \cdot B}$

This latter equation can be used to bring out the rule for expressing fractional roots as exponents. For example, how do you express the square root of 2 as a power of 2? We seek to find a number x such that $\sqrt{2} = 2^x$. Squaring both sides of this equation gives $2 = (2^x)^2$ or $2^1 = 2^{2x}$. This tells us that $1 = 2x$ and therefore, $x = 1/2$. We conclude that the square root of a number A equals $A^{1/2}$. Similarly the cube root of A equals $A^{1/3}$, the twelfth root of A is $A^{1/12}$, etc.

From the given exercises the students should obtain a familiarity with:

1. The words: exponent and base
2. The laws of exponents:
   a. $A^x \cdot A^y = A^{x+y}$
   b. $A^x \div A^y = A^{x-y}$
   c. $(A^x)^y = A^{xy}$
   d. $(A^x)^{1/y} = A^{x/y}$.

B. Operating with **Bases** 2, 3 and 5

After reviewing exponents, let three different groups in the class develop tables to the bases 2, 3 and 5.

Examples:

$2^1 = 2$	$3^1 = 3$	$5^1 = 5$
$2^2 = 4$	$3^2 = 9$	$5^2 = 25$
$2^3 = 8$	$3^3 = 27$	$5^3 = 125$
$2^4 = 16$	$3^4 = 81$	$5^4 = 625$
$2^5 = 32$	$3^5 = 243$	$5^5 = 3,125$
$2^6 = 64$	$3^6 = 729$	$5^6 = 15,625$
$2^7 = 128$	$3^7 = 2,187$	$5^7 = 78,125$
$2^8 = 256$	$3^8 = 6,561$	$5^8 = 390,625$
$2^9 = 512$	$3^9 = 19,683$	$5^9 = 1,953,125$
$2^{10} = 1,024$	$3^{10} = 59,049$	$5^{10} = 9,765,625$
$2^{11} = 2,048$	$3^{11} = 177,147$	$5^{11} = 48,828,125$
$2^{12} = 4,096$	$3^{12} = 531,441$	$5^{12} = 244,140,625$
$2^{13} = 8,192$	$3^{13} = 1,594,323$	$5^{13} = 1,220,703,125$
$2^{14} = 16,384$	$3^{14} = 4,782,969$	$5^{14} = 6,103,515,625$
		$5^{15} = 30,517,578,125$

Let the students make up and solve problems similar to the following:

Examples: a. $19,683 \times 243 = 3^9 \times 3^5 = 3^{14} = 4,782,969$

b. $\frac{16,384}{512} = \frac{2^{14}}{2^9} = 2^5 = 32$

c. $(243)^2 = (3^5)^2 = 3^{10} = 59,049$

d. $\sqrt{15,625} = (5^6)^{1/2} = 5^3 = 125$

At this point, the students should appreciate the ease and speed of performing operations by exponents.

3.2 Logarithms

Now we are ready to introduce the idea of logarithms as exponents. The instructor may write the following relationships. (Explain that the word logarithm is usually written as log.)

$2^3 = 8$	$\log_2 8 = 3$
$2^5 = 32$	$\log_2 32 = 5$
$2^7 = 128$	$\log_2 128 = 7$
$2^{11} = 1,024$	$\log_2 1024 = 11$
$2^6 = 64$	$\log_2 64 = 6$
$2^4 = 16$	$\log_2 16 = 4$

Through discussion and questions the students should grasp the idea that the base being used is base 2. Questions should arise sooner or later on the definition for the logarithm of a number. Ask the students to give a definition for the logarithm of a number based on what has been discussed. List some of the definitions on the board. It is hoped that from the definitions we will get something similar to:

Definition: The logarithm of a number to base 2 is the exponent to which 2 must be raised in order to obtain the given number.

Examples: $4^3 = 64$, so 3 is the logarithm of 64 to the base 4

$5^2 = 25$, so 2 is the logarithm of 25 to the base 5

$10^2 = 100$, so 2 is the logarithm of 100 to the base 10

So we write: if $6^2 = 36$, then $2 = \log_6 36$

if $10^3 = 1000$, then $3 = \log_{10} 1000$

Let us look at the table below:

A	B
1	3
2	9
3	27
4	81
5	243
6	729
7	2,187
8	6,561
9	19,683
10	59,049
11	177,147

Ask the students if they can tell what the relatiohship is between column A and column B. It is hoped that they will discover that the nth term in A is n and the nth term of the number in B is 3^n, hence the numbers in column A are the logarithms of those in column B to the base 3.

Let the students multiply several pairs of numbers in column B. Here it is hoped that they will discover that to multiply two numbers in B, you need only add the corresponding numbers in A. The answer is found in column B opposite the sum of the numbers added in A.

Example: Take 27 x 729. Opposite 27 we see 3; opposite 729 we see 6. 6 + 3 = 9. Opposite 9 read 19,683 in the B column. This the product of 27 and 729.

Ask the students to do the reverse of what they just did:

Take 729 ÷ 27. Opposite 729 we have 6 and opposite 27 we see 3. 6 - 3 = 3. Opposite 3 read 27 in the B column. This is the division of 729 by 27.

This shows that in effect two numbers can be multiplied by adding their logarithms, and that one number may be divided into another by substracting its logarithm from the logarithm of the other.

Exercises: Let the students make up and solve problems similar to the ones above using other bases (2 and 5).

In an effort to give students a feeling for the rule for taking logarithms of powers of numbers, have them examine and complete tables such as the following:

$\log_{10} 2 = .3010$	$\log_{10} 3 = .4771$	$\log_2 8 = 3$	$\log_3 9 =$ _____
$\log_{10} 4 = .6030$	$\log_{10} 9 =$ _____	$\log_2 64 =$ _____	$\log_3 81 =$ _____
$\log_{10} 8 = .9030$	$\log_{10} 27 = 1.4313$	$\log_2 8^3 =$ _____	$\log_3 9^3 =$ _____
$\log_{10} 16 =$ _____	$\log_{10} 81 =$ _____	$\log_2 8^4 =$ _____	$\log_3 9^4 =$ _____
$\log_{10} 32 =$ _____	$\log_{10} 243 = 2.3855$	$\log_2 8^5 =$ _____	$\log_3 9^5 = 10$

They may also be able to discover from this exercise the rule for taking the logarithm of the root of a number.

In the course of working out the examples the students should be able to write down the fundamental rules of logarithms:

1. $\log (A \times B) = \log A + \log B$
2. $\log (A/B) = \log A - \log B$
3. $\log (A^n) = n$ times $\log A$
4. $\log (\sqrt[n]{A}) = \log A$ divided by n

Exercises: Use the above tables to evaluate the following:

1. 1024×32
2. $\frac{177,147}{19,683}$
3. $(5^3)^4$
4. $(59049)^{1/2}$

3.3 Common Logarithms

The logarithms that we use in practical work such engineering, surveying, and astronomy are to base 10 because the calculations using that base are just as simple as the decimal system is in arithmetic. Unless stated otherwise, the logarithms in the remainder of this section use base 10.

Ask the students to discuss the following chart using the base 10:

$10^0 = 1$

$10^a = 4 \rightarrow 0 < a < 1$

$10^1 = 10$

$10^b = 37 \rightarrow 1 < b < 2$

$10^2 = 100$

$10^c = 442 \rightarrow 2 < c < 3$

$10^3 = 1000$

$10^d = 5850 \rightarrow 3 < d < 4$

In this base, log 10 is 1, log 100 is 2, log 1000 is 3 and so on. Based on the chart ask the students what can they say about the log of a number between 1 and 10, between 10 and 100, and between 100 and 1000. They should discover the following:

1. The log of any number between 1 and 10 is a number between 0 and 1, i.e., a decimal.
2. The log of any number between 10 and 100 is a number between 1 and 2, i.e., 1 plus a decimal.
3. The log of any number between 2 and 3, i.e., 2 plus a decimal.

In general, the form of the logarithm of any number A > 1 is a whole number plus a decimal. The whole number part of the logarithm of A is always one less than the number of digits which precede the decimal point in A. We call this whole number the "characteristic" of the logarithm. The other part (the fractional part) of the logarithm we call the "mantissa".

Example: log 100 = 2.00000. The characteristic of log 100 is 2 and its mantissa is .00000.

Give the students the number 20. Ask them to estimate the logarithm of it. Based on the given information, they should know that its logarithm lies between 1 and 2. At this period the instructor may introduce the students to tables of logarithms. Let the students find the logarithm of 20 in the table. They will see that it is 1.30103. This means $10^{1.30103} = 20$. Ask the students to find the log of 2, 200, and 2000.

Exercise: Find the logarithms of 31, 3.1, 3100 and 31,000.

Antilogarithms

After the students have worked with the logarithm table finding the logarithms of numbers, give them a logarithm and tell them to find the number that corresponds with it.

Give the students the following exercises:

$\log A = 1.78533 \quad \log B = 1.74819 \quad \log C = .84510$

Find A, B and C.

At this point the instructor should introduce the word antilogarithm.

Interpolation

Students should be able at this point to find the mantissa for 3 digit numbers. Ask them to find the log of 3729. The teacher might want to give this example:

$$\left\{\begin{array}{rl} & \log 3730 = 3.5717 \\ 10 \quad 9 & \left\{\begin{array}{l} \log 3729 = \quad ? \\ \log 3720 = 3.5705 \end{array}\right\} d \end{array}\right\} .0012$$

$$\frac{d}{.0012} = \frac{9}{10}, \quad d = .0011$$

$$\log 3729 = 3.5705 + .0011 = 3.5716$$

The same procedure can be used to find N if $\log N = 3.4090$, and .4090 is not in the table.

Ask the students if it is possible to write all numbers as powers of a common base.

Let us take a look at one of the powers of 2; $2^{10} = 1024$. 1024 is very close to being 1,000. Suppose we make that approximation, what can we say? $2^{10} \simeq 1{,}000 = 10^3$

Now, to solve this equation for 2 in terms of powers of 10, we take the 10th root of both sides; of course, the 10th root of the 10th power of 2 is right back to 2 again. But what about the right hand side?

$$2 = (2^{10})^{1/10} = (10^{3})^{1/10} = 10^{\cdot 3}$$

In other words, we have found a way to approximately express 2 as a power of ten. This might be a very useful tool if we were interested in getting an estimate of some very high power of 2.

Exercise: *Make an estimate of* 2^{100}.

Solution: We can replace 2 by $10^{\cdot 3}$ and take the 100th power of it.

$$2^{100} = (10^{\cdot 3})^{100} = 10^{30}$$

So, approximately, 2^{100} is 1 followed by 30 zeros.

Exercise: *Is it possible to express 3 as a power of ten also?*

HINT: Use what we already know about 2 as a power of 10.

Note on the Logarithm Table (Common or Briggsian 4-place)

A knowledge of interpolation will not be necessary to use these tables. They are meant to serve computational needs, to which, in our opinion, they are very well adapted.

3.4 Illustrative Examples

We end this section on logarithms with examples which illustrate the use of the laws of logarithms in computations.

1. The logarithm of a product equals the sum of the logarithms.

Example: 3.458 x 9.446 = ?

From the table of mentissas* we have

log 3.458 = 0.5388

log 9.446 = 0.9753.

Adding these logs gives us 1.5141, and since the antilog of 1.5141 is 32.67, we conclude that 3.458 x 9.446 = 32.67.

2. The log of the quotient of two numbers equals the log of the dividend minus the log of the divisor.

Example: $\frac{3.458 \times 9.446}{1.989 \times 5.34}$

log 3.458 = 0.5388		log 1.989 = 0.2987	
log 9.446 = <u>0.9753</u>		log 5.34 = <u>0.7275</u>	
Sum of logs:	1.5141	Sum of logs:	1.0262

Hence, log of dividend = 1.5141

log of divisor = <u>1.0262</u>

Substracting gives 0.4879 which is the log of the quotient.

The antilog of 0.4879 = 3.075 and $\frac{3.458 \times 9.446}{1.989 \times 5.34} = 3.075$

3. The log of a power of a positive number equals the log of the number multiplied by the exponent.

Example: Use 2 above to evaluate $\left(\frac{3.458 \times 9.446}{1.989 \times 5.34}\right)^5$.

From our work above, we need only evaluate $(3.075)^5$. By our law, $\log (3.075)^5 = 5 \log 3.075$:

$5 \log 3.075 = 5 \times 0.4879 = 2.3395.$

* This work is taken from a four place log table with computed proportional parts. A table such as the one used may be found in the <u>Handbook of Chemistry and Physics</u>, or in some mathematics books, such as <u>Schaum's First Year College Mathematics</u>.

The antilog of 2.3395 = 218.6, therefore

$$\left(\frac{3.458 \times 9.446}{1.989 \times 5.34}\right)^5 = 218.6.$$

4. The log of a root of a positive number equals the log of the number divided by the index of the root.

Example: $\sqrt[4]{\left(\frac{3.459 \times 9.446}{1.989 \times 5.34}\right)^5}$

Taking the log of the power in 3 above and dividing by 4 gives 2.3395/4 = 0.5849.

Antilog 0.5849 = 3.845, which is the required fourth root.

EXERCISES

Use logarithms to perform the following computations:

1. 62.4×17.6
2. $521.9/.821$
3. $(\sqrt[3]{15.7})(8.72)$
4. $\sqrt[5]{16/4.62}$
5. $721/.007$
6. $\dfrac{(32.6)(.421)}{601}$

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IV. THE SLIDE RULE AS A USEFUL TOOL

This section is not intended to replace or supplant the many excellent books already in print on how to use and/or teach the slide rule. Any good slide rule that is purchased has with it instructions on its use and these can be utilized with good effect. This section of the unit is designed primarily to present teachers with a method of introducing their students to the mechanics of the slide rule and to increase the students' ability to manipulate it effectively.

4.1 Making An Adding Machine

Give students paper strips-graph paper ruled to 10 x 10 to the inch is preferred -- and have them to construct a number line. Now students are challenged to use their number lines as adding machines by asking questions such as: *Can you add numbers with these strips?*

Can you subtract?

Will your arrangement work for both positive and negative numbers?

By using two 12-inch rulers, one inverted, the teacher can demonstrate how to add numbers up to 12 plus 12. Now ask the class: *what could be used as a machine to add larger numbers?* If it is not suggested by a student, the teacher can suggest the use of two meter sticks. Now ask the question:

What size numbers can be added by using the two meter sticks?

It should be obvious that by using longer and longer scales we could get a machine to add any two numbers -- but it would be unwieldly and unpractical. The students should be able to understand this.

Suppose we wanted to multiply two numbers such as 16 x 32, *can we do it with the machine we have?* It soon becomes clear that with a "uniform scale" we are able to perform multiplication. Note, however, that these two numbers are both powers of 2: $2^4 \times 2^5$ which equals 2^9. A table of the powers of two makes problems like this very easy. *Complete the following table:*

Power of 2	Number	Power of 2	Number
12		2	4
11		1	2
10		0	1
9		-1	1/2
8	256	-2	
7		-3	1/8
6		-4	
5	32	-5	
4		-6	
3	8	-7	

Use the table to work the following problems:

a. $128 \times 4 = 2^? \times 2^? = 2^? =$

b. $512 \div 32 = 2^? \div 2^? = 2^? =$

(Others may may be added) c. $256 \times .25 \times 16 = 2^? \times 2^? \times 2^? =$

d. $64 \div 256 \times 32 = 2^? \div 2^? \times 2^? =$

In each case the solution of the problem involves the adding or subtracting of exponents (numbers) — these operations are what we have been doing with our paper adding machine. If students are not familiar with the fundamental operations of numbers in exponential form, some time must be devoted to the laws of exponents.

Suppose we make a paper multiplying machine based on the powers of 2. Our paper machine will show both the number and the corresponding power of two as below:

1/32	1/16	1/8	1/4	1/2	1	2	4	8	16	32	64	128	256	512	1024
-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10

With two of these scales extended as far as desired in both directions, it is possible to multiply any two numbers which are powers of two by adding the corresponding powers of 2. We have made a machine which will do multiplication. Our machine is, however, severly limited as it works only for those numbers which are powers of 2.

We would like a machine which will multiply any two numbers; a machine based upon the powers of 10. We can make such a machine by expressing the numbers from 1 to 10 as approximate powers of 10. For example: $2^{10} = 1024$, which is very close to 1000. Suppose we make that approximation, that is: $2^{10} = 1000 = 10^3$

Now to solve this equation for 2 in terms of powers of 10, we take the 10th root of both sides. The 10th root of the 10th power of 2 is 2 itself. *What about the right side?*

$$2 = (2^{10})^{1/10} = (10^3)^{1/10} = 10^{3(1/10)} = 10^{3/10} = 10^{.300}$$

We have found a way to express 2 as an approximate power of 10. *What other numbers can we get from this discovery?*

$$4 = 2(2) = 10^{.300} \cdot 10^{.300} = 10^{.300 + .300} = 10^{.600}$$

$$8 = 2(4) = 10^{.300} \cdot 10^{.600} = 10^{.300 + .600} = 10^{.900}$$

What about the other numbers?

Since 3^9 is almost equal to 20,000, suppose we assume that it is 20,000?

$$3^9 = 20{,}000 = 2 \cdot 10^4 = 10^{.300} \cdot 10^4 = 10^{4.300}$$

Now if we take the 9th root of both sides we get:

$$3 = (3^9)^{1/9} = (10^{4.300})^{1/9} = 10^{\frac{4.300}{9}} = 10^{.478}$$

From this result and our value for 2 as a power of 10 what numbers can we get?

$$6 = 2 \cdot 3 = 10^{.300} \cdot 10^{.478} = 10^{.778}$$

$$9 = 3 \cdot 3 = 10^{.478} \cdot 10^{.478} = 10^{.956}$$

In similar fashion we compute 5 and 7 as powers of 10. $5 = 10^{.700}$

$7 = 10^{.848}$

We are now ready to construct our **multiplying** machine based on the values:

$1 = 10^{0}$ $\quad\quad$ $6 = 10^{.778}$

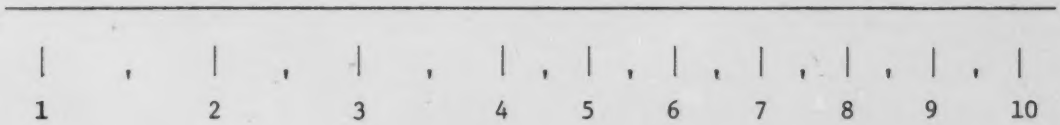
$2 = 10^{.300}$ $\quad\quad$ $7 = 10^{.848}$

$3 = 10^{.478}$ $\quad\quad$ $8 = 10^{.900}$

$4 = 10^{.600}$ $\quad\quad$ $9 = 10^{.956}$

$5 = 10^{.700}$ $\quad\quad$ $10 = 10^{1}$

Take two strips of paper 10 inches long and label the points as follows:



1 is at the left end of the scale

2 is .300 of the length from the left end

3 is .478 of the length from the left end

4 is .600 of the length from the left end

5 is .700 of the length from the left end

6 is .778 of the length from the left end

7 is .848 of the length from the left end

8 is .900 of the length from the left end

9 is .956 of the length from the left end

10 is 1.00 of the length from the left end

As any number can be expressed in scientific notation as $M \times 10^{n}$, where $1 \leq M \leq 9$ and n is an integer, we can use the machine we have made to multiply any two numbers. The students should now be challenged to use their paper machines to multiply several pairs of numbers. After this the instructor can introduce the slide rule as a more sophisticated version of their machine.

The following sections have been effectively used in teaching students how to manipulate the slide rule.

4.2 Parts of the Slide Rule

There are three parts to the slide rule: (1) the body, (2) the slide, and (3) the indicator with a hairline. The indicator is sometimes referred to as the cursor. Have the students name and identify each part on their individual rules.

Look at your rules and identify the scales found there. How are these scales named? Are the scales all different? No, two scales are identical. *Which scales are exactly the same?* C and D. The scales on the slide and body work together in solving problems. The slide with its hairline aids us in reading our answer. The most important scales are the two which are identical, the C and D.

Handling of the slide rule is important. The following points for the care of the slide rule are given by Clyde B. Clason in Delights of the Slide Rule:

1. Hold the rule by the end nearest to the setting when you move the indicator. This will pinch the slide and body together slightly and prevent a shift in the setting.
2. Always hold the body part by its edges and push or pull the slide by its ends.
3. Keep your fingers away from the edges of the slide: you may work dirt into the grooves.

4. Grasp the indicator by its edges and near the hairline; then you will not interfere with the tensioning spring when you move it.

Clason also offers the following "don'ts":

1. Don't leave your slide rule out when you are through using it. Put it back in its case and the case in a safe place.
2. Don't use it for anything but calculations. It is not a measuring device, nor a straight edge for ruling lines.
3. Don't drop it.
4. Don't spill hot coffee on it. Upsetting a water glass over it is not a good idea either.
5. Don't throw away the printed literature that comes with your slide rule. It would not even hurt to read it.

4.3 How to Read Numbers on the Slide Rule

1. The use of a demonstration slide rule will be very useful in teaching students how to locate and read numbers on the slide rule.

Use the D scale. Have students to locate the primary divisions starting from the left index and continuing to the right indices. It should be noted that the numbers are closer together as one progresses from left to right. *How many primary divisions are there on the slide rule?* 10

Now have the students locate and read aloud the primary divisions beginning at the left index which we will call 1. A student can set the numbers on the demonstration rule as they are said aloud: 1, 2, 3, ..., 9, 10. Note that the right index is read as ten.

Have the students begin again at the left index, which we now call 10, and read the primary divisions as 10, 20, 30, ..., 90, 100.

Repeat the cycle reading the left index as 100, 1000, 10,000. Be certain that students read the right index correctly. These numbers could have been written on the blackboard in a horizontal line as they were found, each series under the preceding one, as follows:

1	2	3	4	5	6	7	8	9	10
10	20	30	40	50	60	70	80	90	100
100	200	300	400	500	600	700	800	900	1000

What can you say about the location of numbers which begin with the same digit?

Where would we find 5 million? .5? .05? .0005?

How do these numbers differ? Hopefully, someone will realize that they differ by powers of ten. The teacher can now explain that the slide rule is a device for adding (or subtracting) powers of 10.

2. Now, beginning at the left index again, ask students how many major divisions are between primary 1 and primary 2. *Since there are 10 divisions, what does each division represent?* Answers will vary -- what they represent depends on what we are calling the left index. Let us call these the secondary divisions. If we call the left index 1, each secondary division is .1. Left index 10, each secondary division is 1. Left index .1, each secondary division is .01. Left index 1000, each secondary division is 100.

Be sure that the students understand that each secondary division is one-tenth of what we call the left index and this is true for the entire length of the rule for that particular reading.

Have students read the secondary divisions aloud, writing the numbers on the board, calling the left index 1. We get 1.1, 1.2, 1.3, ..., 2.1, 2.2, 2.3, ..., 9.7, 9.8, 9.9, 10.0. Repeat with the left index as 10, 100, etc., writing the numbers under the previous series 11, 12, 13, ..., 21, 22, 23, ..., 97, 98, 99, 100, 110, 120, 130, ..., 210, 220, 230, ..., 970, 980, 990, 1000.

As a result of their experience in reading the primary divisions, students will recognize that numbers which begin with the same two non-zero digits and whose other digits are zeroes are located in the same place. Emphasize that the first digit of a number is read from the primary divisions and the second digit is read from the secondary divisions. Give the students plenty of practice in locating and reading numbers with two significant digits, with and without decimals.

3. Now raise the question, *where do you think we will find the third digit of a three-digit number, for example,* 152? Have students try to find this number on their slide rule. When someone locates it, have it placed on the demonstration rule for all to see. Now ask:

Why did you choose that position?

How many divisions are there between secondary 5 *and* 6?

What is the value of each division?

What will be the value of each of these divisions if the secondary divisions are tenths? (left index as 1) .01 are ones? (left index as 10) .1 are tens? (left index as 100) 1 are hundreds? (left index as 1000) 10.

We call these tertiary divisions and use them to locate the third digit of a number. Call the left index 10, how do we read the number at the very first graduation? No, not 11; we won't have 11 until we reach a secondary division. How many divisions before we reach 11? (10). Then each division must be what? (1/10 or .1) Therefore, this number is? 10.1 Have the student to read each tertiary graduation up to primary 2.

How many tertiary divisions are there between 20 and 21? Of course, 2 and 3 are closer together and the lines would be too crowded. Call the left index 10 and have the students read the divisions as 20.2, 20.4, 20.6, ..., 21.2, 21.4, 21.6, ..., 204, 206, ..., 212, 214, 216, ..., 296, 298, 300.

Repeat the cycle between 30 and 40, 300 and 400. What is happening to the graduations? Right, they are becoming closer and closer together.

If we call the left index 100, the secondary divisions between 400 and 500 have what value? Locate 410 on your slide rule. What other numbers are at that same point? 4.1, 41, 41,000, 4100, etc. What number will we find halfway between 400 and 410? (405). How many tertiary divisions are between 400 and 410? Therefore, each division must count as? .5. Now, read each division between 400 and 500 as 405, 410, 415, 420, ..., 500. Call the left index 10 and read each division between 40 and 50 as 40.5, 41.0, 41.5, 42.0, ..., 50.0.

What about the divicsions between 500 and 600? They are closer together, but are read in the same way, i.e., 505, 510, 515, ..., 600.

Have students locate the following:

242 - 2 primary, 4 secondary, 2 tertiary

485 - 4 primary, 8 secondary, 5 tertiary

605 - 6 primary, 0 secondary, 5 tertiary

178 - 1 primary, 7 secondary, 8 tertiary

545 - 3 primary, 4 secondary, 5 tertiary

376 - 3 primary, 7 secondary, 6 tertiary

Emphasize that

first digit is always primary

second digit is always secondary

third digit is always tertiary

We always consider the largest digit in each division to the left of the hairline. Now have students locate the numbers 109, 560, 283, 397, 901, 477, and others where the third digit is between two graduations.

4. Some numbers that we want may contain more than three digits, like 2545. *How would we locate this number? Pass the 2 on the primary, pass the 5 on the tertiary. What are the 2 closest numbers to 2545 that end in zero?* 2540, 2550. *Where is 2545 in relation to these two numbers? Halfway between. Now locate* 2545.

In similar fashion have students locate other 4-digit numbers. You may prepare a list of numbers to be located, both large ones and numbers with decimals.

4.4 Multiplying on the Slide Rule (C and D Scales)

1. Multiplying 1 - digit Numbers

Pick some easy 1 - digit numbers to which the answers are already known; have students place the left index on the C scale over one of the factors on the D scale, move the indicator to the other factor on the C scale and read the answer under the hairline on the D scale. Have students find 2x2 = _____, 2x3 = _____, 2x4 = _____, 2x5 = _____, 2x6 = _____(This causes a problem because the 6 on the slide is off the rule.) What can we do? It will be suggested, eventually, that we put the right index on the 2. Now we can do 2x7 = _____, 2x8 = _____, 2x9 = _____ 2x10 = _____. Try other 1-digit numbers. Repeat the process with a 1-digit number times a 2 and 2-digit number, being certain to include some where the right index must be used.

2. Multiplying 2-digit Numbers

Repeat the above process with a 2-digit number times a 2-digit number. Reading the answer now becomes very important. There are two methods for determining the answer, the common sense method and the use of scientific notation. The common sense method is adequate in most cases involving only two numbers. For example: What is the product of 29 and 63. We read 183 under the hairline. The answer must be close to 30 x 60 which is 1800, a 4-digit number. Therefore, our answer must contain 4-digits, then it must be 1830 and not 183. Suppose it was 2.9 x 63 - close to 3 x 6 - answer is 18.3; it was .29 x 6.3 - close to .3 x 6 - answer is 1.83; it was .29 x .63 - close to .3 x .6 - answer is .183.

How accurate is our answer? If we compute 29 x 63 by hand we get 1827, a difference of 3. Our error is only 3/1827 or .00164. This

is less than .002 or .2% - a very small error, accurate enough for most computations.

Students must be given enough practice in multiplying and reading the answer correctly until they have confidence in their ability. Occasionally have them do the same problem with paper and pencil and compare the two answers. Then have them compute the percentage of error - which should always be less than 1% - and thus gain confidence in the accuracy of their answers.

4.5 Dividing on the Slide Rule (C and D Scales)

1. First be sure that students understand that division is the inverse of multiplication. Have students set the slide rule to show that $2 \times 3 = 6$. Point out that we start at 2 on the D scale; move the indicator to 3 on the C scale, and find the answer under the hairline on the D scale. Point out that by reversing these steps we get the result 2.

   It will be helpful here to introduce the concept of scientific notation for the benefit of students who have not previously studied it or who have forgotten it. Division, especially of very large or very small numbers, becomes easier in this form. Thus the division of $4.8 \times 10^{2} / 1.2 \times 10^{6} = 4.0 \times 10^{-4} = 0.0004$. This is easier for most students than 480/1,200,000. Using scientific notation can help to ease fears about the location of the decimal point. Now have students begin dividing numbers for which they know the answers; pointing out that the answers are always on the D scale, under one of the indexes on the C scale.

It should be emphasized to the students that the slide rule gives digits only; the operator has to decide what the number is by correctly placing the decimal point. Dividing 42 by 74 gives the digits, 568. Knowing that the answer must be less than 1 and more than 1/2, the confident operator writes the answer as .568 and not 5.68 or .0568.

2. After students gain confidence in their ability to handle division problems have them begin to do combined operations. It is best to begin with examples such as $\frac{67 \times 48}{85}$. Point out to students that it is easier to divide and multiply alternately. Thus, in the above example we should divide 67 by 85 and then multiply the results by 48. Steps used are:

    *1. Move hairline to 67 on D scale.

    2. Move slide so that 85 on C scale is under hairline.

    3. Move hairline to 48 on C scale.

    4. Read answers on D scale, under hairline.

    *Not really necessary, but helps in locating first number. Observe that the hairline and slide are moved alternately. Incidentally, our example $\frac{67 \times 48}{85}$ gives the digits 379. Since $\frac{48}{85}$ would give an answer slightly greater than 1/2, our answer must be slightly more than 1/2 times 67; that is 37.9.

Write the step-by-step procedure in solving:

$$\frac{4.2 \times 8.8 \times 3.9}{13 \times 2.2}$$

1. Move hairline to 42 on D scale.
2. Move slide so that 13 is under hairline.
3. Move hairline to left index, necessary because 88 is "off scale."
4. Move right index under hairline.
5. Move hairline to 8.8 on C scale.
6. Move slide so that 22 is under hairline.
7. Move hairline to 3.9 on C scale.
8. Read answer 5.03 on D scale.

Have students work several examples in which there is one more factor in the numerator than the denominator. This helps to instill in them the idea of alternate moves of the hairline and slide.

In examples where the difference in the number of factors in the numerator and denominator is not one, such as:

$$\frac{28 \times 45.6 \times 93}{59} \quad \text{or} \quad \frac{29 \times 43}{15.5 \times 38 \times 98}$$

additional factors (1) may be added as needed:

$$\frac{28 \times 45.6 \times 93}{59 \times 1} \quad \text{or} \quad \frac{29 \times 43 \times 1 \times 1}{15.5 \times 38 \times 98}$$

This permits them to continue to follow the pattern of hairline-slide-hairline, etc.

4.6 Raising to Powers and Extracting Roots (A and K Scales)

Tell students that the A and D scales are used for squares and square roots; the K and D scales for cubes and cube roots and let them experiment and discover how to use them. You may wish to supervise a special-interest group outside of the regular class time in working out difficult slide rule computations.

References

A Teaching Guide for Slide Rule Instruction, Nartung, Maurice L.

How to Use the Slide Rule, Bishop, Calvin

Simplifying the Slide Rule - Marks

Delights of the Slide Rule - Clason, Clyde B.

"The Slide Rule" - Dorothy S. Hogan, Bishop College

"Paper Computing Machine - CRG

A series of film loops on the slide rule is available (Ealing)